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On a Minimal Property of Cardinal and Periodic Lagrange Splines

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1. INTRODUCTION

In [7], Reimer shows that the Lagrange splines $L_j := L(\cdot - j)$ form an "extremal" basis for the space $\mathbb{S}_n$ of bounded cardinal splines of a given odd degree n in the sense that

$$\|L\|_\infty = 1. \tag{1}$$

Here, $\mathbb{S} = \mathbb{S}_n$ consists of all bounded splines of degree n with simple knots, at the integers Z when n is odd, and at the half-integers $\frac{1}{2} + Z$ when n is even. Further, L is the unique element of $\mathbb{S}$ which satisfies $L(j) = \delta_{0j}$, all j.

In [9], Siepmann and Sündermann obtain (1) for the *cubic* case, $n = 3$, as a limiting consequence of the fact that, for any N,

$$s^{(N)}(x) := \sum_{1}^{N} (L_j^{(N)}(x))^2 \leqslant 1, \tag{2}$$

with $L_j^{(N)} := L^{(N)}(\cdot - j)$ the N-periodic cardinal spline which satisfies $L_j^{(N)}(k) = \delta_{kj}$, $k = 1, ..., N$. Further, they conjecture that (2) holds for arbitrary odd degree.

We observe here that, for any odd or even degree,

$$s^{(N)} = \sum_{j=1}^{N} |Sf_j|^2/N, \tag{3}$$

with Sf_j the cardinal spline interpolant (at the integers) to the complex exponential

$$f_j(x) := \exp(2\pi ijx/N). \tag{4}$$

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This implies (2), hence (1) in the limit, since

$$|Sf_j| \leqslant 1, \tag{5}$$

as shown in Schoenberg [8] for odd degree and, with much more effort, in de Boor [1] for even degree. In fact, it is shown in [1] that $|Sf_j(x)|$ decreases strictly monotonically away from the integers (at which it has the value 1, of course) to a minimum value at the point halfway between the integers. Hence (3) allows the same description for the function $s^{(N)}$ and so confirms that the figure for $s^{(N)}$ drawn for the cubic case in [9] is qualitatively correct for all degrees. In particular, equality in (5) or (2) only holds at the interpolation points and nowhere else.

It follows, more precisely than (1), that

$$-1 < L(x) \leqslant 1, \qquad \text{with equality iff } x = 0. \tag{6}$$

2. The Main Result

We begin with a simple expression for $L_k^{(N)}$ in terms of the cardinal spline interpolant to the complex exponential:

$$L_k^{(N)} = \frac{1}{N} \sum_{j=1}^{N} \exp(-2\pi ikj/N)\, Sf_j, \qquad \text{all } k. \tag{7}$$

Since both sides are cardinal splines, we merely need to show that they agree at the integers. But this is obvious since Sf_j agrees with $f_j = \exp(2\pi ij/N)$ at the integers, hence

$$\sum_{j=1}^{N} \exp(-2\pi ikj/N)(Sf_j)(m) = \sum_{j=1}^{N} \exp(2\pi i(m-k)\,j/N) = \begin{cases} 1, & \text{if } m-k \in NZ \\ 0, & \text{otherwise.} \end{cases}$$

Formula (7) was first obtained by Golomb [3], for odd n and even N, by a somewhat different route.

Since $|Sf_j| \leqslant 1$, this gives immediately that

$$-1 < L_k^{(N)}(x) \leqslant 1, \qquad \text{with equality iff } x \in k + NZ. \tag{8}$$

We now come to the main result.

Proposition.

$$s^{(N)} = \sum_{k=1}^{N} (L_k^{(N)})^2 = \frac{1}{N} \sum_{j=1}^{N} |Sf_j|^2.$$

Proof. Formula (7) implies that

$$s^{(N)} = \frac{1}{N^2} \sum_{j=1}^{N} \sum_{m=1}^{N} \left\{ \sum_{k=1}^{N} \exp(-2\pi i(j+m)k/N)\, Sf_j Sf_m \right\} = \frac{1}{N} \sum_{j=1}^{N} Sf_j Sf_{N-j}.$$

The proposition follows from this with the aid of the facts

$$Sf_{j+N} = Sf_j, \qquad Sf_{-j} = \overline{Sf_j}$$

which are inherited from the corresponding properties of the f_j's. ▮

COROLLARY. $0 < s^{(N)}(x) \leqslant 1$, *with equality iff* $x \in Z$.

Proof. The proposition together with the fact that $|Sf_j| \leqslant 1$ gives the inequalities. The fact that equality does not occur for nonintegral x follows from the fact that $L^{(N)}(x) \neq 0$ for nonintegral x (see Richards [5]).

3. THE CARDINAL ANALOGUE

The N-periodic Lagrange spline $L^{(N)}$ is an infinite folding of the Lagrange spline,

$$L^{(N)}(x) = \sum_{k=-\infty}^{\infty} L(x+kN),$$

with the sum absolutely convergent, because of the exponential decay of L. The exponential decay also ensures therefore that

$$L(x) = \lim_{N \to \infty} L^{(N)}(x), \qquad \text{uniformly on compact sets.}$$

The preceding results therefore imply that

$$|L(x)| \leqslant 1 \text{ and } \sum_{j \in Z} (L(x-j))^2 \leqslant 1, \qquad \text{for all } x.$$

Given these inequalities, it follows that equality occurs in the former iff $x=0$ and in the latter iff $x \in Z$. For, otherwise we would have, in either case, some nonintegral x for which $1 = |L(x)| = \lim |L^{(N)}(x)|$, and therefore, by the proposition, $0 = \lim L^{(N)}(y) = L(y)$ for some $Y \equiv x \pmod 1$

This would contradict the fact that L can only vanish at integers (see de Boor and Schoenberg [2, Part C, Sect. 2]).

In summary, we have (6) and the following statement:

The 1-*periodic function* $s := \sum_{j \in Z} (L(\cdot - j))^2$ *equals* $\lim s^{(N)}$, *is symmetric with respect to* $x = \frac{1}{2}$, *achieves its maximum value*, 1, *at* 0, *and its minimum value at* $\frac{1}{2}$, *and is strictly monotone in between.*

The strict monotonicity follows from the monotonicity of $s^{(N)}$ and the fact that each polynomial component of s has a positive leading coefficient, hence s' does not vanish identically on any proper interval.

Finally, there are consequences concerning the norm of the interpolation operator S as a map from sequences to various L_p spaces, leading to generalizations of Richards' result [6] that $\|S\|_{2,2} = 1$. These results have been detailed in the author's thesis [4].

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